JOURNAL OF APPROXIMATION THEORY 83, 347–370 (1995)

Estimates of Best Approximation and Fourier Transforms in Integral Metrics

M. I. GANZBURG*

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012

AND

E. R. LIFLYAND†

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel

Communicated by Rolf J. Nessel

Received March 22, 1994; accepted in revised form January 5, 1995

Under some assumptions on a function F and its Fourier transform $\hat{F}$ we prove new estimates of best approximation of F by entire functions of exponential type σ in $L_p(\mathbb{R})$, $1 \leqslant p < 2$. The proof is based on some inequalities for $\hat{F}$ in $L_1(\mathbb{R})$ which may be treated as generalizations of results of Bausov and Telyakovskii. As an application we obtain exact estimates of best approximation of some infinitely differentiable functions.

1. INTRODUCTION

Let $A_\sigma(F)_p$, $1 \leqslant p \leqslant \infty$, denote the error in approximating to $F \in L_p(\mathbb{R})$ by entire functions of exponential type $\sigma > 0$, i.e.,

$$A_\sigma(F)_p = \inf_{g \in B_\sigma} \|F - g\|_{L_p(\mathbb{R})}$$

* This research was supported in part by the Program for Refugee Scientists in the New York Academy of Sciences. E-mail:ganzbrgm@acfy.nyu.edu.

† This research was supported in part by "Maagara," a special project for absorption of new immigrants, in the Department of Mathematics and Computer Science, Bar-Ilan University. E-mail: liflyand@bimacs.cs.biu.ac.il.

0021-9045/95 $12.00

where B_σ is the set of entire functions of exponential type σ. Here $\mathbb{R}$ means the real axis. Let us put

$$\hat{F}(t)=\hat{F}_c(t)-i\hat{F}_s(t)=\int_0^{\infty} F(x)e^{-ixt}\,dx.$$

In this paper we shall study the rate of $A_\sigma(F)_p$ for some classes of functions. Our initial aim was to find the exact order of decrease of $A_\sigma(\varphi_{\lambda,\alpha})_p$, where

$$\varphi_{\lambda,\alpha}=|x|^\lambda \exp(-A\,|x|^{\alpha}), \qquad \lambda\in\mathbb{R}, \quad \alpha>0, \quad A>0,$$

is the classical infinitely differentiable function. This problem for polynomial approximation in $L_\infty(-1,1)$, $\lambda=0$, $\alpha=2$, was posed by Bernstein more than 40 years ago. A lower estimate of $A_\sigma(\varphi_{\lambda,\alpha})_p$ may be obtained in a standard way [8] but for a long time we could not find the efficient upper estimate. Much attention has been given to upper estimates of $A_\sigma(F)_p$ in the literature. A Jackson-type theorem

$$A_\sigma(F)_p\leqslant C\omega_{k,p}(F,\sigma^{-1}), \tag{1.1}$$

where $\omega_{k,p}(F,t)$ is the integral modulus of smoothness of order $k\geqslant 1$ has been obtained by Bernstein [2] for $p=\infty$, $k=1$, while for $1\leqslant p\leqslant\infty$, $k=2$ the estimate (1.1) has been proved by Akhiezer [1]; A. F. Timan and M. F. Timan [17] have generalized this result to any $k>2$, $1\leqslant p\leqslant\infty$. There are many generalizations of (1.1) in different directions (cf. [16], [13], [6]). This estimate is efficient for some functions of finite smoothness but gives no good results for infinitely differentiable, or analytic functions [7]. Besides, there is no general method for computation of $\omega_{k,p}(F,\sigma^{-1})$, and this problem is very difficult for many individual functions, especially in the case $1\leqslant p<\infty$. For these reasons, in many cases estimates of $A_\sigma(F)_p$, using the Fourier transform of F, are more efficient than (1.1).

The known Markov-type theorem proved by Krein [9] and Nagy [12] makes it possible to find $A_\sigma(F)_1$ for some functions with regularly decreasing $\hat{F}$. In particular, if $F\in L_1(\mathbb{R})$ is a continuous even function and $\hat{F}_c(t)$ is 3-monotonic (that is, each of the first three derivatives preserves a sign) for $t>\sigma_0$, then

$$A_\sigma(F)_1=(8/\pi)\sum_{k=0}^{\infty}(-1)^k\frac{\hat{F}_c((2k+1)\sigma)}{2k+1}, \qquad \sigma>\sigma_0.$$

This theorem is efficient only for very special classes of functions. For instance, $\varphi_{\lambda,\alpha}$ do not satisfy the conditions of the theorem. It follows from

Hausdorff-Young's theorem [20] that for a continuous function $F \in L_1(\mathbb{R})$ such that $\hat{F} \in L_1(\mathbb{R}) \cap L_q(\mathbb{R})$, $q = p/(p-1)$, $2 \leqslant p \leqslant \infty$,

$$A_\sigma(F)_p \leqslant C \left(\int_\sigma^\infty (|\hat{F}(t)|^q + |\hat{F}(-t)|^q)\, dt \right)^{1/q}. \tag{1.2}$$

There is no analogous inequality for $1 \leqslant p < 2$.

The aim of the present paper is to obtain the efficient estimates for $A_\sigma(F)_p$ in the case $1 \leqslant p < 2$.

Our main result is given in the following inequalities which are essentially the basis for other results of the paper:

$$\begin{aligned} A_\sigma(F)_1 &\leqslant \|F - Q_\sigma(F)\|_{L_1(\mathbb{R})} \\ &\leqslant C \left(|\hat{F}_c(\sigma)| + |\hat{F}_c(2\sigma)| + \int_0^\infty t \left| \frac{d^2}{dt^2} \hat{F}_c(\sigma + t) \right| dt \right) \end{aligned} \tag{1.3}$$

where F is an even function satisfying some conditions, and Q_σ is a linear operator of approximation. Using (1.3) and properties of Q_σ we shall obtain an estimate of $A_\sigma(F)_p$ for $1 < p < 2$. These results are stated in Section 3.

The proof of (1.3) is based on new estimates of Fourier transforms in $L_1(\mathbb{R})$, which are proved in Section 2. These results are integral analogues of some inequalities due to Bausov and Telyakovskii [15], and they are interesting in themselves.

At last, as an application of our results, we shall obtain exact upper estimates for best approximation of some infinitely differentiable functions, like $\varphi_{\lambda,\alpha}$. These inequalities are proved in Section 4.

Note that throughout this paper C will denote different positive constants not depending on the essential parameters z, σ, etc., on the variables x, y, N, etc., and on the functions f, F, $\hat{f}$, $\hat{F}$.

2. Estimates of Fourier Transforms

Many different conditions for coefficients of a trigonometric series that yield the integrability of this series are well-known. Among them are the conditions due to Boas-Telyakovskii, Fomin, Sidon-Telyakovskii, C. Stanojevic, Moricz, Buntinas, Tanovic-Miller and others (the lists of references in [4], [11] give a comprehensive bibliography in this field). Different conditions of integrability of Fourier transforms are well-known as well. But those corresponding to the afore-mentioned conditions for series were almost not investigated till recently. Perhaps the paper of Trigub [18] was the first where the systematic study of such relations was begun. In the paper of the second author [10] an integral analogue of Boas-Telyakovskii conditions (see, e.g., [15, (1.2), (1.3)]; these conditions are the strongest in the range of such results) was established as follows (see Corollary 1 in [10]):

THEOREM A. *Let f be a locally absolutely continuous function on* $[0, \infty)$, *and* $\lim_{x \to +\infty} f(x) = 0$. *Then for every* $y > 0$

$$\hat{f}_c(y) = \theta_1 \gamma_1(y) \tag{2.1}$$

$$\hat{f}_s(y) = \frac{1}{y} f\left(\frac{\pi}{2y}\right) + \theta_2 \gamma_2(y) \tag{2.2}$$

where $|\theta_j| \leqslant C$, *and for* $j = 1, 2$

$$\int_0^\infty |\gamma_j(y)|\, dy \leqslant \int_0^\infty |f'(x)|\, dx + \int_0^\infty \left| \int_0^{u/2} \frac{f'(u-x) - f'(u+x)}{x}\, dx \right| du.$$

The following theorem is very close to Theorem A and its proof is strongly based on it.

THEOREM 1. *Let f be a locally absolutely continuous function on* $[0, \infty)$, *and* $\lim_{x \to +\infty} f(x) = 0$. *Then for every* $z > 0$, $y > \pi/2z$

$$\hat{f}_c(y) = \frac{\sin zy}{y} \left(f\left(z - \frac{\pi}{2y}\right) - f\left(z + \frac{\pi}{2y}\right) \right) + \theta \Gamma_1(y) \tag{2.3}$$

where $|\theta| \leqslant C$, *and*

$$\begin{aligned} \int_{\pi/2z}^\infty |\Gamma_1(y)|\, dy \leqslant{} & \int_0^\infty |f'(x)|\, dx \\ & + \int_0^z \left| \int_0^{\min(u/2,\,(z-u)/2)} \frac{f'(u-x) - f'(u+x)}{x}\, dx \right| dx \\ & + \int_0^\infty \left| \int_0^{u/2} \frac{f'(z+u-x) - f'(z+u+x)}{x}\, dx \right| du + |f(z)|. \end{aligned} \tag{2.4}$$

Theorem 1 generalizes another result of Telyakovskii [15, Corollary 1]. Let us postpone the proof of this theorem. We need some auxiliary results, similar to those obtained in [15].

LEMMA 1. *Let g be a locally absolutely continuous function on* $[0, \infty)$. *Then the following inequality holds*

$$\int_0^\infty \left| \int_0^{u/2} \frac{g(u-x) - g(u+x)}{x}\, dx \right| \leqslant \ln 3 \int_0^\infty t\, |g'(t)|\, dt.$$

Proof. We have

$$\int_0^\infty \left| \int_0^{u/2} \frac{g(u-x)-g(u+x)}{x}\,dx \right| du$$

$$= \int_0^\infty \left| \int_0^{u/2} \frac{dx}{x} \int_{u-x}^{u+x} g'(t)\,dt \right| du$$

$$\leqslant \int_0^\infty du \int_{u/2}^{3u/2} |g'(t)| \ln \frac{u}{2\,|u-t|}\,dt = \ln 3 \int_0^\infty t\,|g'(t)|\,dt.$$

This completes the proof. ∎

Consider two auxiliary functions

$$\beta(x) = \begin{cases} f(x), & 0 \leqslant x < \dfrac{z}{3}, \\[2mm] \left(2 - \dfrac{3x}{z}\right) f(x), & \dfrac{z}{3} \leqslant x \leqslant \dfrac{2z}{3}, \\[2mm] 0, & x > \dfrac{2}{3}\,z, \end{cases}$$

and

$$\gamma(x) = \begin{cases} f(z-x) - \beta(z-x), & 0 \leqslant x \leqslant z, \\ 0, & x > z. \end{cases}$$

Evidently, $f(x) = \beta(x) + \gamma(z-x)$ on $[0, z]$.

LEMMA 2. *Let f be an absolutely continuous function on $[0, z]$. Then the following inequalities hold*:

$$\int_0^\infty (|\beta'(x)| + |\gamma'(x)|)\,dx \leqslant C\left(\int_0^z |f'(x)|\,dx + |f(z)|\right), \tag{2.5}$$

$$\int_0^\infty \left| \int_0^{u/2} \frac{\beta'(u-x)-\beta'(u+x)}{x}\,dx \right| du + \int_0^\infty \left| \int_0^{u/2} \frac{\gamma'(u-x)-\gamma'(u+x)}{x}\,dx \right| du$$

$$\leqslant C\left(\int_0^z \left| \int_0^{\min(u/2,\,(z-u)/2)} \frac{f'(u-x)-f'(u+x)}{x}\,dx \right| du \right.$$

$$\left. + \int_0^z |f'(x)|\,dx + |f(x)|\right). \tag{2.6}$$

Proof. Let us denote

$$F(x)=\begin{cases}\frac{3}{z}f(x), & \frac{z}{3}\leqslant x\leqslant\frac{2}{3}z,\\ 0, & \text{otherwise.}\end{cases}$$

We are not able to apply Lemma 1 to F immediately, because F may be not absolutely continuous in the neighborhoods of $z/3$ and $(2/3)z$. Let us consider a continuous function $F_\varepsilon(x)$ on $[0,\infty)$ which coincides with F on $[z/3,(2/3)z]$, vanishes outside $[z/3-\varepsilon,(2/3)z+\varepsilon]$ for sufficiently small ε, and is linear on $[z/3-\varepsilon, z/3]$ and $[(2/3)z,(2/3)z+\varepsilon]$. Since F_ε satisfies the conditions of Lemma 1, we obtain

$$\begin{aligned}&\int_0^\infty\left|\int_0^{u/2}\frac{F_\varepsilon(u-x)-F_\varepsilon(u+x)}{x}\,dx\right|du\\ &\leqslant\ln 3\int_0^\infty t\,|F_\varepsilon'(t)|\,dt\\ &\leqslant\ln 3\left\{\frac{3}{z}\int_{z/3}^{(2/3)z}t\,|f'(t)|\,dt+\left|f\left(\frac{z}{3}\right)\right|+\left|f\left(\frac{2}{3}z\right)\right|\right\}\\ &\leqslant\ln 3\left(2\int_{z/3}^{(2/3)z}|f'(t)|\,dt+\left|-\int_{z/3}^{z}f'(t)\,dt+f(z)\right|\right.\\ &\qquad\left.+\left|-\int_{2z/3}^{z}f'(t)\,dt+f(z)\right|\right)\\ &\leqslant 3\ln 3\left(\int_0^z|f'(t)|\,dt+|f(z)|\right).\end{aligned}\tag{2.7}$$

But one can calculate easily that

$$\begin{aligned}&\int_0^\infty\left|\int_0^{u/2}\frac{(F-F_\varepsilon)(u-x)-(F-F_\varepsilon)(u+x)}{x}\,dx\right|du\\ &\leqslant C\left(\left|f\left(\frac{z}{3}\right)\right|+\left|f\left(\frac{2}{3}z\right)\right|\right)\leqslant C\left(\int_0^z|f'(t)|\,dt+|f(z)|\right).\end{aligned}\tag{2.8}$$

Thus we obtain from (2.7) and (2.8) that

$$\int_0^\infty\left|\int_0^{u/2}\frac{F(u-x)-F(u+x)}{x}\,dx\right|du\leqslant C\left(\int_0^z|f'(t)|\,dt+|f(z)|\right).\tag{2.9}$$

Let us denote $B'(x) = \beta'(x) + F(x)$, i.e.,

$$\beta(x) = \begin{cases} f'(x), & 0 \leqslant x < \dfrac{z}{3}, \\ \left(2 - \dfrac{3x}{z}\right) f'(x), & \dfrac{z}{3} \leqslant x \leqslant \dfrac{2z}{3}, \\ 0, & x > \dfrac{2}{3} z. \end{cases} \tag{2.10}$$

We have from (2.9) and (2.10)

$$\int_0^{\infty} \left| \int_0^{u/2} \frac{\beta'(u-x) - \beta'(u+x)}{x} \, dx \right| du \tag{2.11}$$

$$\leqslant \int_0^{\infty} \left| \int_0^{u/2} \frac{B'(u-x) - B'(u+x)}{x} \, dx \right| du + C\left(\int_0^{z} |f'(x)| \, dx + |f(z)| \right).$$

It follows from (2.10) that $B'(u-x) = f'(u-x)$ for $u \leqslant z/3$, and

$$B'(u+x) = \begin{cases} f'(u+x), & x \leqslant \dfrac{z}{3} - u, \\ \left(2 - \dfrac{3(u+x)}{z}\right) f'(u+x), & x > \dfrac{z}{3} - u. \end{cases}$$

Thus,

$$\int_0^{z/3} \left| \int_0^{u/2} \frac{(B' - f')(u-x) - (B' - f')(u+x)}{x} \, dx \right| du$$

$$= \int_0^{z/3} \left| \int_{z/3-u}^{u/3} \left(\frac{3}{z}(u+x) - 1 \right) \frac{f'(u+x)}{x} \, dx \right| du \leqslant \int_0^{z} |f'(x)| \, dx. \tag{2.12}$$

Let $z/3 \leqslant u \leqslant (2/3)z$. Then it follows from (2.10) that

$$B'(u-x) = \begin{cases} \left(2 - \dfrac{3(u-x)}{z}\right) f'(u-x), & x \leqslant u - \dfrac{z}{3}, \\ f'(u-x), & x > u - \dfrac{z}{3}, \end{cases}$$

and

$$B'(u+x)=\begin{cases}\left(2-\dfrac{3(u+x)}{z}\right)f'(u+x), & x<\dfrac{2}{3}z-u,\\[2mm] 0, & x\geqslant\dfrac{2}{3}z-u.\end{cases}$$

This yields

$$\begin{aligned}&\int_{z/3}^{(2/3)z}\left|\int_0^{u/2}\frac{B'(u-x)-B'(u+x)}{x}\,dx\right.\\&\qquad\left.-\left(2-\frac{3u}{z}\right)\int_0^{u/2}\frac{f'(u-x)-f'(u+x)}{x}\,dx\right|du\\&=\int_{z/3}^{(2/3)z}\left|\int_0^{u-z/2}\frac{3f'(u-x)}{z}\,dx+\left(\frac{3u}{z}-1\right)\int_{u-z/3}^{u/2}\frac{f'(u-x)}{x}\,dx\right.\\&\qquad\left.+\int_0^{\min(u/2,\,2z/3-u)}\frac{3f'(u+x)}{z}\,dx+\left(2-\frac{3u}{z}\right)\int_{(2/3)z-u}^{u/2}\frac{f'(u+x)}{x}\,dx\right|du\\&\leqslant\frac{3}{z}\int_{z/3}^{(2/3)z}\left\{\int_0^{u/2}|f'(u-x)|\,dx+\int_0^{u/2}|f'(u+x)|\,dx\right\}du\\&\leqslant\int_0^z|f'(x)|\,dx.\end{aligned}$$

Thus, we obtained that

$$\begin{aligned}&\int_{z/3}^{(2/3)z}\left|\int_0^{u/2}\frac{B'(u-x)-B'(u+x)}{x}\,dx\right|du\\&\qquad\leqslant\int_{z/3}^{(2/3)z}\left|\int_0^{u/2}\frac{f'(u-x)-f'(u+x)}{x}\,dx\right|du+\int_0^z|f'(x)|\,dx.\end{aligned}\tag{2.13}$$

Let $u\geqslant\frac{2}{3}z$. The formula (2.10) gives us that $B'(u+x)=0$ and

$$B'(u-x)=\begin{cases}0, & x\leqslant u-\dfrac{2}{3}z\\[2mm] \left(2-\dfrac{3(u-x)}{z}\right)f'(u-x), & x>u-\dfrac{2}{3}z.\end{cases}$$

Hence

$$\int_{(2/3)z}^{\infty} \left| \int_0^{u/2} \frac{B'(u-x)-B'(u+x)}{x}\,dx \right| du$$
$$= \int_{(2/3)z}^{\infty} \left| \int_{u-(2/3)z}^{u/2} \left(2 - \frac{3(u-x)}{z}\right) \frac{f'(u-x)}{x} \right| du$$
$$\leqslant \frac{3}{z} \int_{(2/3)z}^{4z/3} du \int_{u-(2/3)z}^{u/2} |f'(u-x)|\,dx \leqslant 2 \int_0^z |f'(x)|\,dx. \quad (2.14)$$

Collecting the estimates (2.12)–(2.14), we have

$$\int_0^{\infty} \left| \int_0^{u/2} \frac{B'(u-x)-B'(u+x)}{x}\,dx \right| du$$
$$\leqslant \int_0^{(2/3)z} \left| \int_0^{u/2} \frac{f'(u-x)-f'(u+x)}{x}\,dx \right| du + 4 \int_0^z |f'(x)|\,dx.$$

If $u > z/2$, then

$$\int_{z/2}^{(2/3)z} \left| \int_{(z-u)/2}^{u/2} \frac{f'(u-x)-f'(u+x)}{x}\,dx \right| du$$
$$\leqslant \frac{3}{z} \int_{z/2}^{2z/3} du \int_{(z-u)/2}^{u/2} (|f'(u-x)| + |f'(u+x)|\,dx \leqslant \int_0^z |f'(x)|\,dx.$$

So we have

$$\int_0^{\infty} \left| \int_0^{u/2} \frac{B'(u-x)-B'(u+x)}{x}\,dx \right| du$$
$$\leqslant \int_0^{(2/3)z} \left| \int_0^{\min(u/2,(z-u)/2)} \frac{f'(u-x)-f'(u+x)}{x}\,dx \right| du + 5 \int_0^z |f'(x)|\,dx.$$

Taking into account (2.11) we obtain that the inequality

$$\int_0^{\infty} \left| \int_0^{u/2} \frac{\beta'(u-x)-\beta'(u+x)}{x}\,dx \right| du$$
$$\leqslant \int_0^{(2/3)z} \left| \int_0^{\min(u/2,(z-u)/2)} \frac{f'(u-x)-f'(u+x)}{x}\,dx \right| du$$
$$+ C\left(\int_0^z |f'(x)|\,dx + |f(z)| \right)$$

holds.

Now we have

$$\int_0^\infty |\beta'(x)|\,dx \leqslant \int_0^{z/3} |f'(x)|\,dx + \int_{z/3}^{(2/3)z} \left(2-\frac{3x}{z}\right) |f(x)|\,dx$$

$$+\frac{3}{z}\int_{z/3}^{(2/3)z} |f(x)|\,dx$$

$$\leqslant 2\int_0^z |f'(x)|\,dx + |f(x)|.$$

So we have proved (2.5) and (2.6) for β. The corresponding estimates for γ are similar to those for β. Lemma 2 is proved. ∎

We are able now to prove Theorem 1.

Proof of Theorem 1. We have

$$\int_0^\infty f(x)\cos xy\,dx = \int_0^z f(x)\cos xy\,dx + \int_z^\infty f(x)\cos xy\,dx.$$

After simple calculations we obtain for the last integral

$$\int_z^\infty f(x)\cos xy\,dx = \int_0^\infty f(z+x)\cos(z+x)\,y\,dx$$

$$= \cos zy \int_0^\infty f(z+x)\cos xy\,dx$$

$$-\sin zy \int_0^\infty f(z+x)\sin xy\,dx,$$

and it suffices to apply Theorem A to both integrals.

Furthermore, we have for the interval $[0, z]$

$$\int_0^z f(x)\cos xy\,dx = \int_0^z \beta(x)\cos xy\,dx + \int_0^\infty \gamma(z-x)\cos xy\,dx.$$

Now, we apply (2.1) to the first integral on the right-hand side. And for the second one we have

$$\int_0^z \gamma(z-x)\cos xy\,dx = \int_0^z \gamma(x)\cos(z-x)\,y\,dx$$

$$= \cos zy \int_0^z \gamma(x)\cos xy\,dx + \sin zy \int_0^z \gamma(x)\sin xy\,dx,$$

and again we apply (2.1) to the first integral, (2.2) to the second one. It remains to apply Lemma 2 to the estimates of the remainders obtained.

Theorem 1 is proved. ∎

The following result may be treated as a corollary to Theorem 1 and a generalization of one result of Bausov and Telyakovskii (see the corresponding prototypes for trigonometric series in [15, (3.72)–(3.74)]).

THEOREM 2. *Let f' be a locally absolutely continuous function with* $\lim_{x\to+\infty} f(x)=0$, *and*

$$\int_0^{\infty} x\,|f''(x)|\,dx<\infty.$$

Then for every $z>0$, the relation (2.3) *holds by*

$$\int_{\pi/2z}^{\infty} |\Gamma_1(y)|\,dy \leqslant |f(0)|+|f(z)|+\int_0^{\infty} \frac{x\,|z-x|}{z+x}\,|f''(x)|\,dx. \tag{2.15}$$

In addition,

$$\int_0^{\infty} |\hat{f}_c(x)|\,dx \leqslant C\left(\int_0^{z} \frac{|f(z-x)-f(z+x)|}{x} + |f(0)|+|f(z)|+\int_0^{\infty} \frac{x\,|z-x|}{z+x}\,|f''(x)|\,dx\right). \tag{2.16}$$

Remark 1. The main condition in Theorem 2, that is the integrability of $x\,|f''(x)|$, is the well-known condition of quasi-convexity of the function f (see, e.g., [3, p. 248]). This class of functions play an important role in different branches of analysis.

Proof of Theorem 2. Notice that the conditions of Theorem 2 yield $\lim_{x\to+\infty} f'(x)=0$. Indeed, it is enough to integrate by parts the integral $\int_0^{\infty} xf''(x)\,dx$ and apply simple computations to the result.

The following relation may be verified immediately

$$f(x)=\frac{z-x}{z}\,f(0)+\frac{x}{z}\,f(z)-\frac{z-x}{z}\int_0^{x} tf''(t)\,dt - \frac{x}{z}\int_x^{z}(z-t)\,f''(t)\,dt. \tag{2.17}$$

Therefore, we obtain for each x, $0 \leqslant x \leqslant z$, that

$$|f(x)| \leqslant |f(0)| + |f(z)| + \int_0^z \frac{t(z-t)}{z} |f''(t)| \, dt. \tag{2.18}$$

In order to obtain (2.15) we have to estimate all the terms on the right-hand side of (2.4).

Using Lemma 1 for $g(t) = f'(z+t)$ we have

$$\int_0^\infty \left| \int_0^{u/2} \frac{f'(z+u-t) - f'(z+u+t)}{t} \, dt \right| du \leqslant 2 \ln 3 \int_z^\infty |f''(x)| \, (x-z) \, dx$$
$$\leqslant 2 \ln 3 \int_z^\infty \frac{x(x-z)}{z+x} |f''(x)| \, dx. \tag{2.19}$$

Furthermore, we obtain

$$\int_0^z \left| \int_0^{\min(u/2,\,(z-u)/2)} \frac{f'(u-x) - f'(u+x)}{x} \, dx \right| du$$
$$\leqslant \int_0^z du \int_0^{\min(u/2,\,(z-u)/2)} \frac{dx}{x} \int_{u-x}^{u+x} |f''(t)| \, dt$$
$$= \int_0^{z/4} |f''(t)| \, dt \int_{(2/3)t}^{2t} \ln \frac{u}{2\,|u-t|} \, du$$
$$+ \int_{x/4}^{(3/4)z} |f''(t)| \, dt \int_{(2/3)t}^{z/2} \ln \frac{u}{2\,|u-t|} \, du$$
$$+ \int_{z/4}^{(3/4)z} |f''(t)| \, dt \int_{z/2}^{(2t+z)/3} \ln \frac{z-u}{2\,|u-t|} \, du$$
$$+ \int_{(3/4)z}^{z} |f''(t)| \, dt \int_{2t-x}^{(2t+z)/3} \ln \frac{z-u}{2\,|t-u|} \, du. \tag{2.20}$$

Four inner integrals on the right-hand side of (2.20) may be calculated directly by the integrating by parts. For example, for $z/4 \leqslant t \leqslant z/2$ we have

$$\int_{z/2}^{(2t+z)/3} \ln \frac{z-u}{2(u-t)} \, du = (z/2 - t) \ln 2(1 - 2t/z) + (z-t) \ln \frac{3z}{4(z-t)}. \tag{2.21}$$

The first summand on the right-hand side of (2.21) is negative, the second is less than $(z-t) \ln \frac{3}{2}$. Since $t/(z+t) \geqslant \frac{1}{5}$ for the range of t considered, the integral in (2.21) may be estimated by $C(t\,|z-t|/(z+t))$. Other estimates may be obtained in a similar way.

This yields the following estimate:

$$\int_0^z \left| \int_0^{\min(u/2,\,(z-u)/2)} \frac{f'(u-x)-f'(u+x)}{x}\, dx \right| du$$

$$\leqslant C \int_0^z \frac{t(z-t)}{z+t}\, |f''(t)|\, dt. \tag{2.22}$$

Now we have to estimate $\int_0^\infty |f'(x)|\, dx$.

$$\int_0^z |f'(x)|\, dx = \int_0^{z/2} \left| f'\left(\frac{z}{2}\right) - \int_x^{z/2} f''(t)\, dt \right| dx$$

$$+ \int_{z/2}^z \left| f'\left(\frac{z}{2}\right) + \int_{z/2}^x f''(t)\, dt \right| dx$$

$$\leqslant z \left| f'\left(\frac{z}{2}\right) \right| + \int_0^{z/2} t\, |f''(t)|\, dt + \int_{z/2}^z (z-t)\, |f''(t)|\, dt. \tag{2.23}$$

Differentiating the identity (2.17), we obtain

$$zf'\left(\frac{z}{2}\right) = f(z) + t(0) + \int_0^{z/2} tf''(t)\, dt - \int_{z/2}^z (z-t)\, f''(t)\, dt. \tag{2.24}$$

Furthermore,

$$\int_z^\infty |f'(x)|\, dx = \int_z^\infty dx \left| \int_x^\infty f''(t)\, dt \right| \leqslant \int_z^\infty (t-z)\, |f''(t)|\, dt$$

$$\leqslant 2 \int_z^\infty \frac{t(t-z)}{z+t}\, |f''(t)|\, dt. \tag{2.25}$$

Combining inequalities (2.4), (2.18), (2.19), (2.22)–(2.25), we obtain the estimate (2.15). In order to prove (2.16) we need the following estimates.

$$\int_{\pi/2z}^\infty \left| \frac{\sin zy}{y} \left(f\left(z - \frac{\pi}{2y}\right) - f\left(z + \frac{\pi}{2y}\right) \right) \right| dy$$

$$\leqslant \int_0^z \frac{|f(z-x) - f(z+x)|}{x}\, dx, \tag{2.26}$$

$$\int_0^{\pi/2z}\left|\int_0^{\infty} f(x)\cos xy\,dx\right|dy$$
$$=\int_0^{\pi/2z}\left|\frac{1}{y}\int_0^{\infty} f'(x)\sin xy\,dx\right|dy$$
$$\leqslant\int_0^{\pi/2z}dy\int_0^{\pi/2y}|f'(x)|\,x\,dx+\int_0^{\pi/2z}\left|\frac{1}{y^2}\int_{\pi/2y}^{\infty} f''(x)\cos xy\,dx\right|dy$$
$$\leqslant\frac{\pi}{2}\int_0^{\infty}|f'(x)|\,dx+\frac{\pi}{2}\int_z^{\infty}(x-z)\,|f''(x)|\,dx$$
$$\leqslant C\left(\int_0^{\infty}|f'(x)|\,dx+\int_z^{\infty}\frac{x(x-z)}{z+x}\,|f''(x)|\,dx\right). \tag{2.27}$$

Using (2.3), (2.15), (2.26), (2.27), we complete the proof of Theorem 2. ∎

The following statement provides us with a generalization of Theorem 2 to functions with derivative having a jump discontinuity at one point.

COROLLARY 1. *Let f' be a locally absolutely continuous function on $[0, z)$ and (z, ∞), $z>0$. Suppose, further, that $|f'(z\pm)|<\infty$, $\lim_{x\to+\infty} f(x)=0$, and*

$$\int_0^{z-} x\,|f''(x)|\,dx+\int_{z+}^{\infty} x\,|f''(x)|\,dx<\infty.$$

Then

$$\int_0^{\infty}|\hat{f}_c(x)|\,dx\leqslant C\left(|f(0)|+|f(z)|+\int_0^{z}\frac{|f(z-x)-f(z+x)|}{x}\,dx\right.$$
$$\left.+\int_0^{z-}\frac{x(z-x)}{z+x}\,|f''(x)|\,dx+\int_{z+}^{\infty}\frac{x(x-z)}{z+x}\,|f''(x)|\,dx\right). \tag{2.28}$$

Proof. Let us put for $x\in[z-\varepsilon, z+\varepsilon]$, where $\varepsilon>0$ is small enough,

$$g(x)=\frac{f'(z+\varepsilon)+f'(z-\varepsilon)}{4\varepsilon^2}(x-z)^3+\frac{f'(z+\varepsilon)-f'(z-\varepsilon)}{4\varepsilon}(x-z)^2$$
$$-\frac{f'(z+\varepsilon)+f'(z-\varepsilon)}{4}(x-z)-\varepsilon\,\frac{f'(z+\varepsilon)-f'(z-\varepsilon)}{4}$$
$$-\frac{f(z+\varepsilon)-f(z-\varepsilon)}{4\varepsilon^3}(x-z)^3+3\,\frac{f(z+\varepsilon)-f(z-\varepsilon)}{4\varepsilon}(x-z)$$
$$+\frac{f(z+\varepsilon)+f(z-\varepsilon)}{2},$$

and $g(x)=f(x)$ otherwise. It is evident that g satisfies the conditions of Theorem 2. So we have,

$$\begin{aligned}
\int_0^\infty |\hat{f}_c(x)|\,dx &\leq \int_0^\infty |\hat{g}_c(x)|\,dx + \int_0^\infty |\hat{f}_c(x)-\hat{g}_c(x)|\,dx \\
&\leq C\Bigg(|f(0)|+|f(z)|+\int_0^z \frac{|f(z-x)-f(z+x)|}{x}\,dx \\
&\quad+\int_0^{z-\varepsilon} \frac{x(z-x)}{z+x}\,|f''(x)|\,dx \\
&\quad+\int_{z+\varepsilon}^\infty \frac{x(x-z)}{z+x}\,|f''(x)|\,dx\Bigg)+C\,|g(z)-f(z)| \\
&\quad+C\int_{z-\varepsilon}^{z+\varepsilon} \frac{x\,|z-x|}{z+x}\,|g''(x)|\,dx \\
&\quad+C\int_0^z \frac{|f(z-x)-g(z-x)+g(z+x)-f(z+x)|}{x}\,dx \\
&\quad+\int_0^\infty |\hat{f}_c(x)-\hat{g}_c(x)|\,dx \\
&= CI_1(\varepsilon)+CI_2(\varepsilon)+CI_3(\varepsilon)+CI_4(\varepsilon)+I_5(\varepsilon).
\end{aligned}$$

We have $I_1(\varepsilon)\leq I_1(0)$, and $CI_1(0)$ coincides with the right-hand side of the inequality (2.28). It now remains to prove that $\lim_{\varepsilon\to 0} I_j(\varepsilon)=0$, $j=2, 3, 4, 5$.

It is easy to see, that

$$\begin{aligned}
I_2(\varepsilon) &= \left|\frac{f(z+\varepsilon)+f(z-\varepsilon)}{2}-\varepsilon\,\frac{f'(z+\varepsilon)-f'(z-\varepsilon)}{4}-f(z)\right| \\
&\leq \frac{1}{2}\varepsilon \sup_{x\in[0,\infty)} |f'(x)|+\frac{1}{2}\,|f(z+\varepsilon)-2f(z)+f(z-\varepsilon)|,
\end{aligned}$$

and the first term tends to zero with $\varepsilon\to 0$ and the second one is small by virtue of the continuity of f.

$$\begin{aligned}
I_3(\varepsilon) &= \int_{z-\varepsilon}^{z+\varepsilon} \frac{x\,|z-x|}{z+x}\left|\frac{f'(z+\varepsilon)+f'(z-\varepsilon)}{2\varepsilon^2}\,3(x-z)\right. \\
&\quad\left.+\frac{f'(z-\varepsilon)-f'(z-\varepsilon)}{2\varepsilon}-\frac{f(z+\varepsilon)-f(z-\varepsilon)}{2\varepsilon^3}\,3(x-\varepsilon)\right|\,dx \\
&\leq 4\varepsilon \sup_{x\in[0,\infty)} |f'(x)|+|f(z+\varepsilon)-f(z-\varepsilon)|,
\end{aligned}$$

and the same reasoning is true.

$$I_4(\varepsilon) \leqslant \int_0^{\varepsilon} \frac{|f(z-x)-f(z+x)|}{x}\, dx + \int_0^{\varepsilon} \left| \frac{f'(z+\varepsilon)+f'(z-\varepsilon)}{4\varepsilon^2} x^3 \right.$$

$$+ \frac{f'(z+\varepsilon)-f'(z-\varepsilon)}{4\varepsilon} x^2 - \frac{f'(z+\varepsilon)+f'(z-\varepsilon)}{4} x$$

$$\left. - \frac{f(z+\varepsilon)-f(z-\varepsilon)}{4\varepsilon^2} x^3 + 3\, \frac{f(z+\varepsilon)-f(z-\varepsilon)}{4\varepsilon} x \right| \frac{dx}{x},$$

and the first integral is small because $\int_0^z ((|f(z-x)-f(z+x)|)/x)\, dx$ converges and the second one may be estimated as $I_3(\varepsilon)$.

Finally, it remains to estimate $I_5(\varepsilon)$. Let us denote $h(x)=f(x)-g(x)$. Notice that $\operatorname{supp} h \subset [z-\varepsilon, z+\varepsilon]$, h is a differentiable function on $[0, \infty)$, and

$$\operatorname{Var}_{[0,\infty)} h' \leqslant \int_{z-\varepsilon}^{z-} |f''(x)|\, dx + \int_{z+}^{z+\varepsilon} |f''(x)|\, dx$$

$$+ 8 \sup_{x\in[0,\infty)} |f'(x)| < C \qquad (2.29)$$

where C does not depend on ε. Thus, $\hat{h}_\varepsilon$ is an entire function of exponential type $z+\varepsilon$, and we are able to use the Wiener inequality [19, p. 81]

$$I_5(\varepsilon) = \int_0^{\infty} |\hat{h}_\varepsilon(x)|\, dx \leqslant \frac{C}{z+\varepsilon} \sum_{n=-\infty}^{\infty} \left| \hat{h}_\varepsilon\left(\frac{n}{z+\varepsilon}\right)\right|. \qquad (2.30)$$

It follows from (2.29) that for $|n|>0$

$$\left|\hat{h}_\varepsilon\left(\frac{n}{z+\varepsilon}\right)\right| = \frac{(z+\varepsilon)^2}{|n|} \left| \int_0^{\infty} h'((z+\varepsilon)x) \sin nx\, dx \right|$$

$$= \frac{(z+\varepsilon)^2}{|n|} \left| \int_0^{2\pi} h'((z+\varepsilon)x) \sin nx\, dx \right| \leqslant Cn^{-2}, \qquad (2.31)$$

where C does not depend on ε and n.

$$+ \frac{f'(z-\varepsilon)+f'(z-\varepsilon)}{2\varepsilon} - \frac{f(z+\varepsilon)-f(z-\varepsilon)}{2\varepsilon^3} 3(x-\varepsilon) \Bigg| dx$$

$$\leqslant 4\varepsilon \sup_{x\in[0,\infty)} |f'(x)| + |f(z+\varepsilon)-f(z-\varepsilon)|,$$

Denoting by $N=[\varepsilon^{-1/2}]$ and using (2.30)–(2.32), we obtain

$$I_5(\varepsilon)\leqslant C\left(\sum_{n=-N}^{N}\left|\hat{h}_c\left(\frac{n}{z+\varepsilon}\right)\right|+\sum_{|n|>N}\left|\hat{h}_c\left(\frac{n}{z+\varepsilon}\right)\right|\right)$$
$$\leqslant C\left((2N+1)\varepsilon+\sum_{|n|>N}n^{-2}\right)\leqslant C\sqrt{\varepsilon}.$$

This completes the proof of Corollary 1. ∎

3. Estimates of Best Approximation

Let F be a function satisfying the following conditions:

(1) F is a continuous function on $\mathbb{R}$;

(2) $F\in L_1(\mathbb{R})$, $\hat{F}_c\in L_1(\mathbb{R})$;

(3) $\hat{F}_c(x)$ and $(d/dx)\,\hat{F}_c(x)$ are locally absolutely continuous functions on $[0,\infty)$ such that

$$\int_0^\infty x\left|\frac{d^2}{dx^2}\hat{F}_c(x)\right|dx<\infty.$$

By S we denote the subset of $L_1(\mathbb{R})$ of all even F satisfying the conditions (1)–(3).

Let us consider an operator $Q_\sigma\colon S\to B_\sigma$

$$Q_\sigma(F,y)=\frac{2}{\pi}\int_0^\sigma(\hat{F}_c(x)-\hat{F}_c(2\sigma-x))\cos xy\,dx.$$

Theorem 3. *If $F\in S$, then*

$$A_\sigma(F)_1\leqslant\|F(\cdot)-Q_\sigma(F,\cdot)\|_{L_1(\mathbb{R})}$$
$$\leqslant C\left(|\hat{F}_c(\sigma)|+|\hat{F}_c(2\sigma)|+\int_0^\infty x\left|\frac{d^2}{dx^2}\hat{F}_c(x+\sigma)\right|dx\right). \qquad (3.1)$$

Proof. Let us denote

$$f(x)=\begin{cases}\hat{F}_c(2\sigma-x), & 0\leqslant x\leqslant\sigma;\\ \hat{F}_c(x), & x>\sigma.\end{cases}$$

We have

$$|f'(\sigma\pm)|<\infty,\qquad f(\sigma)=\hat{F}_c(\sigma),\qquad f(0)=\hat{F}_c(2\sigma). \qquad (3.2)$$

It is clear that f satisfies all the conditions of Corollary 1 for $z=\sigma$. Using (2.28) and (3.2) we obtain

$$\int_0^\infty |\hat{f}_c(x)|\,dx \leqslant C\left(|\hat{F}_c(\sigma)| + |\hat{F}_c(2\sigma)| + \int_0^\sigma \frac{x(\sigma-x)}{\sigma+x}\left|\frac{d^2}{dx^2}\hat{F}_c(2\sigma-x)\right| dx\right.$$
$$\left. + \int_0^\infty \frac{x(x-\sigma)}{\sigma+x}\left|\frac{d^2}{dx^2}\hat{F}_c(x)\right| dx\right)$$
$$\leqslant C\left(|\hat{F}_c(\sigma) + |\hat{F}_c(2\sigma)| + \int_0^\infty x\left|\frac{d^2}{dx^2}\hat{F}_c(x+\sigma)\right| dx\right.$$
$$\left. + \int_0^\infty (x-\sigma)\left|\frac{d^2}{dx^2}\hat{F}_c(x)\right|\right)$$
$$\leqslant C\left(|\hat{F}_c(\sigma)| + |\hat{F}_c(2\sigma)| + \int_0^\infty x\left|\frac{d^2}{dx^2}\hat{F}_c(x+\sigma)\right| dx\right).$$

In order to prove (3.1) it is enough to observe that conditions (1) and (2) imply the identity

$$F(x) - Q_\sigma(F, x) = \frac{2}{\pi}\hat{f}_c(x).$$

Theorem 3 is proved. ∎

The following result is a generalization of Theorem 3 to the case of approximation in $L_p(\mathbb{R})$, $1 \leqslant p \leqslant 2$.

COROLLARY 2. *If $F \in S \cap L_2(\mathbb{R})$, then for $1 \leqslant p \leqslant 2$*

$$A_\sigma(F)_p \leqslant \|F(\cdot) - Q_\sigma(F,\cdot)\|_{L_p(\mathbb{R})} \leqslant C\left(\int_\sigma^\infty (\hat{F}_c(x))^2\,dx\right)^{1-1/p}$$
$$\times\left(|\hat{F}_c(\sigma)| + |\hat{F}_c(2\sigma)| + \int_0^\infty x\left|\frac{d^2}{dx^2}\hat{F}_c(\sigma+x)\right| dx\right)^{(2/p)-1}. \quad (3.3)$$

Proof. Observe that if $F \in L_2(\mathbb{R})$, then

$$\|F(\cdot) - Q_\sigma(F,\cdot)\|^2_{L_2(\mathbb{R})} = C\left(\int_\sigma^\infty (\hat{F}_c(x))^2\,dx + \int_\sigma^\infty (\hat{F}_c(2\sigma-x))^2\,dx\right)$$
$$\leqslant C\int_\sigma^\infty (\hat{F}_c(x))^2\,dx. \quad (3.4)$$

Now, using Hölder's inequality we have

$$\|F(\cdot)-Q_\sigma(F,\cdot)\|^p_{L_p(\mathbb{R})}=\int_R |F(x)-Q_\sigma(F,x)|^{2-p}\,|F(x)-Q_\sigma(F,x)|^{2p-2}\,dx$$
$$\leqslant \|F(\cdot)-Q_\sigma(F,\cdot)\|^{2-p}_{L_1(\mathbb{R})}\,\|F(\cdot)-Q_\sigma(F,\cdot)\|^{2p-2}_{L_2(\mathbb{R})}. \tag{3.5}$$

Combining (3.1), (3.4), (3.5) we complete the proof of Corollary 2. ∎

Remark 2. Periodic analogues of Theorem 3 and Corollary 2 are established in [8].

4. Estimate of Best Approximation of Some Infinitely Differential Functions

Let us put

$$\varphi_{\lambda,\alpha,0}(x)=\begin{cases} x^\lambda\exp(-Ax^{-\alpha}), & x>0\\ 0, & x\leqslant 0,\end{cases}$$

$$\varphi_{\lambda,\alpha,i}(x)=(\operatorname{sgn} x)^i\,|x|^\lambda\exp(-A\,|x|^{-\alpha}),\qquad i=1,2,$$

where $\alpha>0$, $A>0$, $\lambda\in\mathbb{R}$ are some constants.

THEOREM 4. *If* $1\leqslant p\leqslant 2$, $\sigma>0$, $i=0,1,2$, *then*

$$A_\sigma(\varphi_{\lambda,\alpha,i})_p\leqslant C\sigma^{m_p}\exp(-M\sigma^{\alpha/(1+\alpha)}), \tag{4.1}$$

where

$$m_p=-\frac{2\lambda p+\alpha p+2}{2p(1+\alpha)},\qquad M=(1+\alpha^{-1})(A\alpha)^{1/(1+\alpha)}\cos\frac{\alpha\pi}{2(1+\alpha)}. \tag{4.2}$$

For the proof of the theorem we need several auxiliary results. We first will find the asymptotic behavior of the Fourier transform of $\varphi_{\lambda,\alpha,i}$ for $\lambda<-1$. Then we will extend this result to any λ.

Let $\mathbb{C}_+$ denote the complex plane, cut along the negative real axis; let z^μ be the branch of this function in $\mathbb{C}_+$ which takes positive values for real $z>0$.

LEMMA 3. *If $\lambda < -1$, then for $y \to +\infty$*

$$\int_0^{\infty} z^{\lambda} \exp(-Az^{-\alpha} - izy)\, dz$$

$$= Cy^{-(2\lambda+2+\alpha)/2(1+\alpha)}$$

$$\times \exp(-(1+\alpha^{-1})(A\alpha)^{1/(1+\alpha)} y^{\alpha/(1+\alpha)} e^{i\alpha\pi/2(1+\alpha)})(1+o(1)). \quad (4.3)$$

Proof. Let us denote

$$z_0 = (A\alpha)^{1/(1+\alpha)} y^{-1/(1+\alpha)} e^{-i\pi/2(1+\alpha)}$$

$$l_R = \{z \in \mathbb{C}_+ : z = \rho z_0,\ 0 < \rho \leqslant R\}, \qquad R > 1$$

$$\Gamma_R = \left\{ z \in \mathbb{C}_+ : |z| = R, -\frac{\pi}{2(1+\alpha)} < \arg z < 0 \right\}$$

$$D = l_R \cup [0, R] \cup \Gamma_R.$$

The function $f(z) = z^{\lambda} \exp(-Az^{\alpha} - izy)$ is analytic inside the curve D, and $\lim_{R \to +\infty} R \max_{\Gamma_R} |f(z)| = 0$. Therefore, we obtain

$$\int_D f(z)\, dz = 0, \qquad \lim_{R \to +\infty} \int_{\Gamma_R} |f(z)|\, |dz| = 0. \quad (4.4)$$

We have that $\int_{l_R} f(z)\, dz$ coincides with the right-hand side of (4.3). This fact for $\lambda = 0$, $A = 1$ was proved in [5], formula (7.42). Note that there are several misprints in the formula but the proof is correct. For $\lambda < -1$, $A \neq 1$ the proof is analogous. Therefore, (4.4) implies (4.3) and Lemma 1 is proved. ∎

In the case $\lambda > -1$ the functions $\varphi_{\lambda, \alpha, i}$ do not belong to $L_1(\mathbb{R})$, and we shall use another approach.

LEMMA 4. *We have*

$$\varphi^{(m)}_{\lambda, \alpha, 0}(x) = \sum_{j=0}^{m} C_j \varphi_{\lambda - m - \alpha j, \alpha, 0}(x) \quad (4.5)$$

where C_j, $0 \leqslant j \leqslant m$, $m = 0, 1, \ldots$ are some constants.

The identity (4.5) may be verified easily by induction on m.

LEMMA 5. *Let $\lambda \in \mathbb{R}$ and $k = [|\lambda|] + 2$. Then there exists an entire function g_0 of exponential type 1 such that $\|\varphi_{\lambda,\alpha,0} - g_0\|_{L_1(\mathbb{R})} < \infty$, and for all $y > 1$, $m \geqslant k$*

$$\int_{\mathbb{R}} (\varphi_{\lambda,\alpha,0}(x) - g_0(x)) e^{-ixy}\, dx = i^m y^{-m} \int_{\mathbb{R}} \varphi^{(m)}_{\lambda,\alpha,0}(x) e^{-ixy}\, dx. \tag{4.6}$$

Proof. It follows from (4.5) that $\varphi^{(m)}_{\lambda,\alpha,0} \in L_1(\mathbb{R})$ for all $m = k, k+1, \ldots$. In virtue of Jackson's theorem [16, p. 260], [6] there exists a kernel $G_k \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ such that a convolution $g_0(x) = (\varphi_{\lambda,\alpha,0} * G_k)(x)$ belongs to B_1, $g_0^{(m)} \in L_2(\mathbb{R})$, $m \geqslant k$, and

$$\|\varphi^{(m)}_{\lambda,\alpha,0} - g_0^{(m)}\|_{L_1(\mathbb{R})} \leqslant C\, \|\varphi^{(k+m)}_{\lambda,\alpha,0}\|_{L_1(\mathbb{R})}, \qquad m = 0, 1, \ldots. \tag{4.7}$$

Integrating by parts and using the Wiener-Paley theorem [14, p. 13] we have for $y > 1$, $m \geqslant k$

$$\begin{aligned}
&\int_{\mathbb{R}} (\varphi_{\lambda,\alpha,0}(x) - g_0(x)) e^{-ixy}\, dx \\
&\qquad = (-i)^m y^{-m} \int_{\mathbb{R}} (\varphi^{(m)}_{\lambda,\alpha,0}(x) - g_0^{(m)}(x)) e^{-ixy}\, dx \\
&\qquad = i^m y^{-m} \int_{\mathbb{R}} \varphi^{(m)}_{\lambda,\alpha,0}(x) e^{-ixy}\, dx.
\end{aligned}$$

This proves Lemma 5. ∎

Now we are able to prove Theorem 4.

Proof of Theorem 4. Let us put

$$F(x) = \begin{cases} \varphi_{\lambda,\alpha,2}(x), & \lambda < -3, \\ \varphi_{\lambda,\alpha,2}(x) - (g_0(x) + g_0(-x))/2, & \lambda \geqslant -3, \end{cases}$$

where g_0 is the same function as in Lemma 5. For $\sigma \geqslant 1$, $1 \leqslant p \leqslant 2$ we have

$$A_\sigma(F)_p = A_\sigma(\varphi_{\lambda,\alpha,2})_p. \tag{4.8}$$

It follows from (4.3), (4.5) that for $\lambda < -3$, $0 \leqslant l \leqslant 2$, $y > 0$

$$\left|\frac{d^l \hat{F}_c(y)}{dy^l}\right| \leqslant \left|\int_{\mathbb{R}} \varphi_{\lambda+l,\alpha,0}(x) e^{-ixy}\, dx\right| \leqslant C y^{m_1 - l/(1+\alpha)} \exp(-M y^{\alpha/(1+\alpha)}) \tag{4.9}$$

where m_p and M are defined by (4.2).

In the case $\lambda \geqslant -3$, $0 \leqslant l \leqslant 2$, $y > 1$ we obtain from (4.3), (4.5), (4.6)

$$\begin{aligned}\left|\frac{d^l \hat{F}_c(y)}{dy^l}\right| &\leqslant \left|\int_{\mathbb{R}} x^l(\varphi_{\lambda, \alpha, 0}(x) - g_0(x)) e^{-ixy}\, dx\right| \\ &= (1+o(1))\, y^{-(k+2)} \left|\int_{\mathbb{R}} x^l \varphi^{(k+2)}_{\lambda, \alpha, 0}(x) e^{-ixy}\, dx\right| \\ &= (1+o(1))\, y^{-(k+2)} \left|\int_{\mathbb{R}} \varphi_{\lambda + l - (k+2)(1+\alpha), \alpha, 0}(x) e^{-ixy}\, dx\right| \\ &\leqslant C y^{m_1 - l/(1+\alpha)} \exp(-M y^{\alpha/(1+\alpha)}). \qquad (4.10)\end{aligned}$$

Thus, (4.9), (4.10) yield the relation

$$\left|\frac{d^l \hat{F}_c(y)}{dy^l}\right| \leqslant C y^{m_1 - l/(1+\alpha)} \exp(-M y^{\alpha/(1+\alpha)}), \qquad y > 1, \quad 0 \leqslant l \leqslant 2. \quad (4.11)$$

It follows from (4.11) that F satisfies all the conditions of Corollary 2. It remains to estimate the integrals on the right-hand side of the inequality (3.3).

Using (4.11) and the asymptotics ($\mu > 0$, $t \to +\infty$), which may be obtained by integrating by parts twice,

$$\int_t^\infty x^{\beta - 1} e^{-\mu x}\, dx = \mu^{-1} t^{\beta - 1} e^{-\mu t}\left(1 + \frac{\beta - 1}{\mu t} + o(t^{-2})\right)$$

we have

$$\left(\int_\sigma^\infty (\hat{F}_c(x))^2\, dx\right)^{1 - 1/p} \leqslant C\sigma^{(2m_1 + 1/(1+\alpha))(1 - 1/p)} e^{-2M(1-1/p)\sigma}, \quad (4.12)$$

$$\begin{aligned}\int_0^\infty x \left|\frac{d^2 \hat{F}_c(x+\sigma)}{dx^2}\right| dx &= \int_\sigma^\infty (y - \sigma) \left|\frac{d^2 \hat{F}_c(y)}{dy^2}\right| dx \\ &\leqslant C\left(\int_{\sigma^{\alpha/(1+\alpha)}}^\infty y^{m_1(1+\alpha)/\alpha + 1} e^{-My}\, dy - \sigma \int_{\sigma^{\alpha/(1+\alpha)}}^\infty y^{m_1(1+\alpha)/\alpha - 1/\alpha} e^{-My}\, dy\right) \\ &\leqslant C\sigma^{m_1} e^{-M\sigma^{\alpha/(1+\alpha)}}. \qquad (4.13)\end{aligned}$$

Collecting now (3.3), (4.11)–(4.13) we complete the proof of the inequality (4.1) for $i = 2$.

In order to prove (4.1) for $i=0$, and $i=1$ we observe that

$$\begin{aligned} \varphi'_{\lambda,\alpha,1}(x) &= \lambda\varphi_{\lambda-1,\alpha,2}(x) - \alpha\varphi_{\lambda-1-\alpha,\alpha,2}(x); \\ \varphi_{\lambda,\alpha,0}(x) &= (\varphi_{\lambda,\alpha,1}(x) + (\varphi_{\lambda,\alpha,2}(x))/2. \end{aligned} \tag{4.14}$$

Applying (4.14) and (4.1) for $i=2$ we obtain

$$\begin{aligned} A_\sigma(\varphi_{\lambda,\alpha,1})_p &\leqslant C\sigma^{-1}A_\sigma(\varphi'_{\lambda,\alpha,1})_p \leqslant C\sigma^{m_p}e^{-M\sigma^{\alpha/(1+\alpha)}}; \\ A_\sigma(\varphi_{\lambda,\alpha,0})_p &\leqslant \tfrac{1}{2}(A_\sigma(\varphi_{\lambda,\alpha,1})_p + A_\sigma(\varphi_{\lambda,\alpha,2})_p) \leqslant C\sigma^{m_p}e^{-M\sigma^{\alpha/(1+\alpha)}}. \end{aligned}$$

Theorem 4 is proved. ∎

Remark 3. The inequalities (4.1) holds also in the case $2<p\leqslant\infty$. It may be proved by using (1.2). Finally, we note that the estimates (4.1) are precise for all p, $1\leqslant p\leqslant\infty$, with respect to the rate of approximation (as $\sigma\to\infty$), i.e.,

$$A_\sigma(\varphi_{\lambda,\alpha,i})_p \geqslant c\sigma^{m_p}\exp(-M\sigma^{\alpha/(1+\alpha)}).$$

These inequalities are established in [8].

References

1. N. I. Akhiezer, "Lectures on the Theory of Approximation," Gostekhizdat, Moscow, 1947 [in Russian]; English translation *in* "Theory of Approximation," Ungar, New York; Constable, London, 1965.
2. S. N. Bernstein, On the best approximation of continuous functions on the whole real axis by means of entire functions of given degree I, *Dokl. Akad. Nauk SSSR* **51**, No. 5 (1946), 327–330. [in Russian]
3. P. Butzer and R. Nessel, "Fourier Analysis and Approximation," Birkhäuser, Basel/Stuttgart, 1971.
4. M. Buntinas and N. Tanović-Miller, New integrability and L^1-convergence classes for even trigonometric series, II, *in* "Approximation Theory," pp. 103–125, Colloquium Mathematics Society János Bolyai, Vol. 58, Kecskemet, Hungary.
5. M. V. Fedoryuk, "Asymptotics: Integrals and Series," Nauka, Moscow, 1987. [in Russian]
6. M. I. Ganzburg, Multidimensional Jackson theorems, *Siberian Math. J.* **22**, No. 2 (1981), 223–231.
7. M. I. Ganzburg, On best approximation of infinitely differentiable functions, *Siberian Math. J.* **32**, No. 5 (1991), 733–749.
8. M. I. Ganzburg, Estimates of best approximation of infinitely differentiable functions, in preparation.
9. M. G. Krein, On the best approximation of continuously differentiable functions on the whole real axis, *Dokl. Akad. Nauk SSSR* **18** (1938), 615–624. [in Russian]
10. E. R. Liflyand, On asymptotics of Fourier transform for functions of certain classes, *Anal. Math.* **19**, No. 2 (1993), 151–168.

11. F. Móricz, On the integrability and L^1-convergence of complex trigonometric series, *Proc. Amer. Math. Soc.* **113** (1991), 53–64.
12. B. S. Nagy, Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen, *Ber. Akad. Wiss. Leipzig* (1939), 91.
13. S. M. Nikolskii, "Approximation of Functions of Several Variables and Embedding Theorems," Nauka, Moscow, 1977. [in Russian]
14. R. E. A. C. Paley and N. Wiener, "Fourier Transforms in the Complex Domain," Amer. Math. Soc., New York, 1934.
15. S. A. Telyakovskii, An estimate, useful in problems of approximation theory, of the norm of a function by means of its Fourier coefficients, *Proc. Steklov Inst. Math.* **109** (1971), 73–109.
16. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Moscow, Fizmatgiz, 1960 [in Russian]; English translation, New York, Macmillan, 1963.
17. A. F. Timan and M. F. Timan, On the relation between the moduli of smoothness of functions defined on the whole real axis, *Dokl. Akad. Nauk SSSR* **113**, No. 5 (1957), 995–997. [in Russian]
18. R. M. Trigub, Multipliers of Fourier series and approximation of functions by polynomials in the spaces C and L, *Soviet Math. Dokl.* **39**, No. 3 (1989), 494–498.
19. N. Wiener, "The Fourier Integral and Certain of its Applications," Dover, New York, 1933.
20. A. Zygmund, "Trigonometric Series, II," Cambridge Univ. Press, Cambridge, UK, 1959.